The Phyics of Charged Particle Therapy:

Phenomenological interaction models in depth and lateral direction (part I)

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What you have to expect from the following lectures...

- Analytical approximations to compute dose / energy loss in depth
  - Some simple math...
  - A lot of „hacking“ by heuristic or fitted parameters
  - Some probability distributions
  - A Bragg-Curve approximation for particles, esp. protons

- Analytical Lateral Scattering Models
  - Multiple Coulomb Scattering
    → “Avoiding“ Molière Theory
  - Gaussian approximation of the scattered distribution

Interaction of Particles with Matter – what you should have learned / know so far…

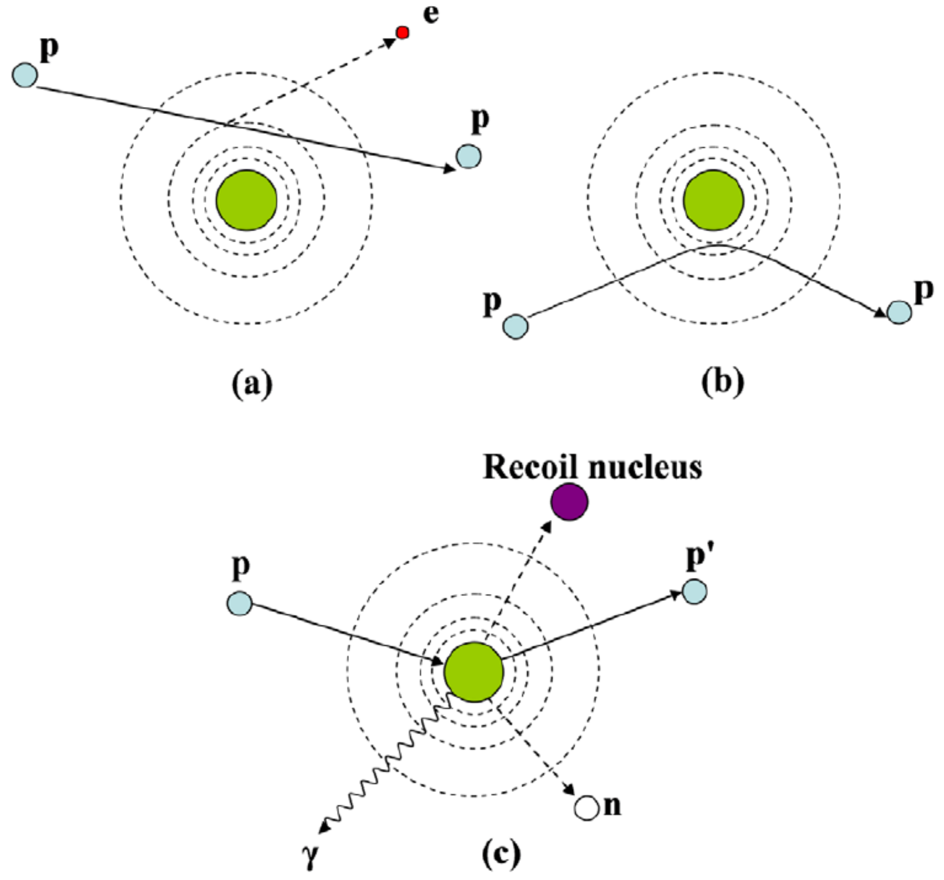
- Fundamental physical interactions

a) Coulomb (inelastic)
→ Energy loss

b) Coulomb (elastic)
→ Deflection

c) Nuclear / Hadronic
→ Secondary Particles
→ Fragmentation ($A > 1$)

d) A tiny bit of Bremsstrahlung…



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Interaction of Particles with Matter – what you should have learned / know so far…

- Important Formulas
  a) Bethe-equation (Mean energy loss):

$$-\left\langle \frac{dE}{dx} \right\rangle = \frac{4\pi}{m_e c^2} \cdot \frac{nz^2}{\beta^2} \cdot \left(\frac{e^2}{4\pi\varepsilon_0} \right)^2 \cdot \left[\ln\left(\frac{2m_e c^2 \beta^2}{I \cdot (1-\beta^2)} \right) - \beta^2 \right] \qquad n = \frac{N_A \cdot Z \cdot \rho}{A \cdot M_u}$$

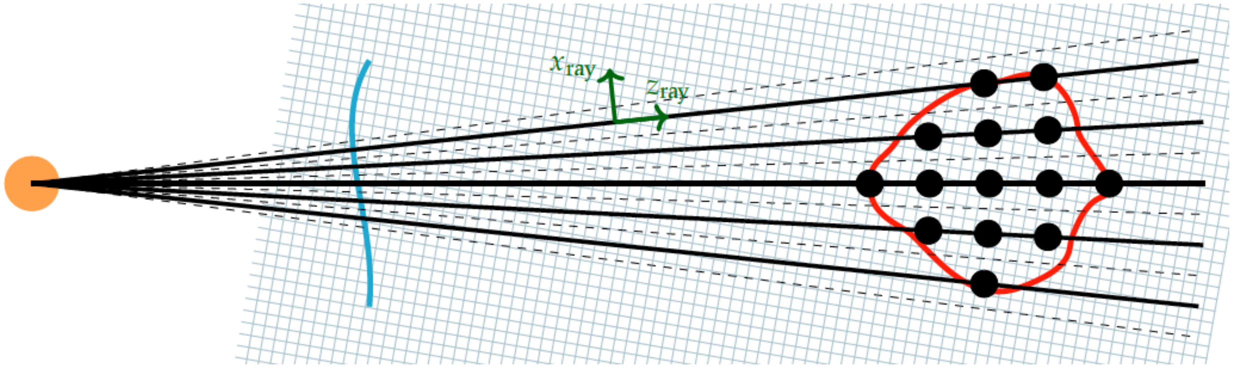
→ Stopping power / (unrestricted) Linear Energy Transfer

  b) Rutherford-Scattering:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{1}{4\pi\varepsilon_0} \frac{Z_1 Z_2 e^2}{4E_0} \right)^2 \frac{1}{\sin^4\left(\frac{\vartheta}{2}\right)}$$

What do we need for treatment planning?

- From CT we know depth of the tumor, but accelerators produce particles with specific energy
  → **Particle Range in Relation to Energy**
- Treatment field build up from many pencil-beams:



→ Fast dose calcution for each individual pencil-beam (vs. Monte Carlo)
→ Can we do some approximations?

The dose in depth

- Dose along incoming direction x:

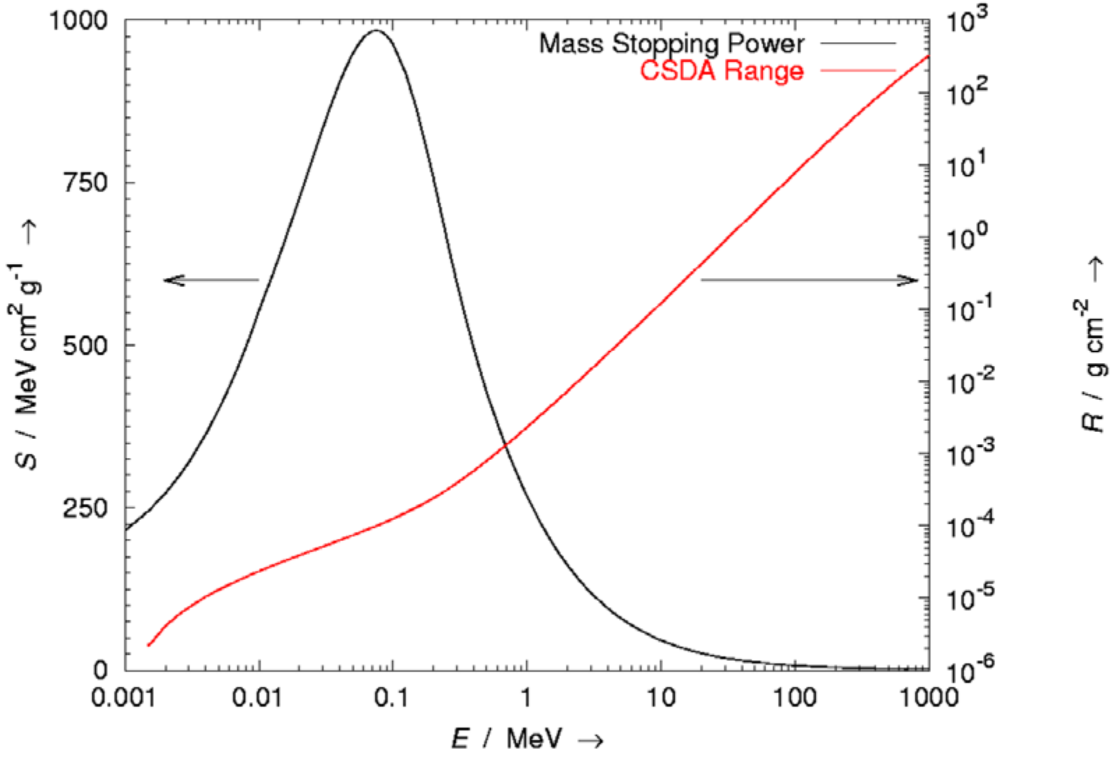
$$D(x) = -\frac{1}{\rho}\frac{d\Psi}{dx} = -\frac{1}{\rho}\left(\Phi(x)\frac{dE}{dx} + \gamma\frac{d\Phi}{dx}E(x)\right)$$

  - Ψ: Energy fluence
  - Φ: Particle fluence
  - γ: Correction for non-local energy loss, ≈ 0.6 for protons

- We need $E(x), \Phi(x), \frac{dE(x)}{dx}$ and $\frac{d\Phi}{dx}$

Range - The Continuous Slowing Down Approximation (CSDA)

- Used to compute the particle range
- Assumes the particle continuous loses energy with dE/dx along x.
  → Integration of stopping power / numerically
- Almost linear on double-logarithmic scale for therapeutic energies
  → Approximate Power-law
- Range-energy conversion:
  $$R(E) \approx \alpha E^p$$
  - $\alpha \propto \sqrt{A_{eff}}$
- Other Ions: $R(E) \approx \frac{A}{Z^2} \alpha E^p$



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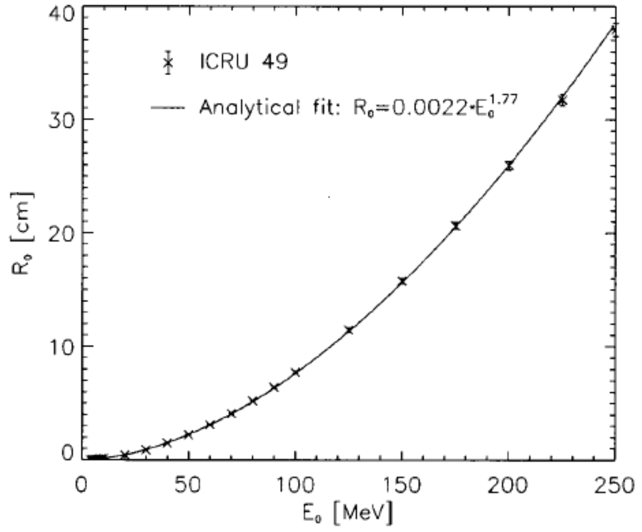
Range-energy conversion

$$R(E) \approx \alpha E^{p}$$

- Proton in water fit by Bortfeld (1996), 10 – 250 MeV:
  - $\alpha \approx 0.022$ ($\propto \sqrt{A_{eff}}$)
  - $p \approx 1.77$



Bortfeld (1996)

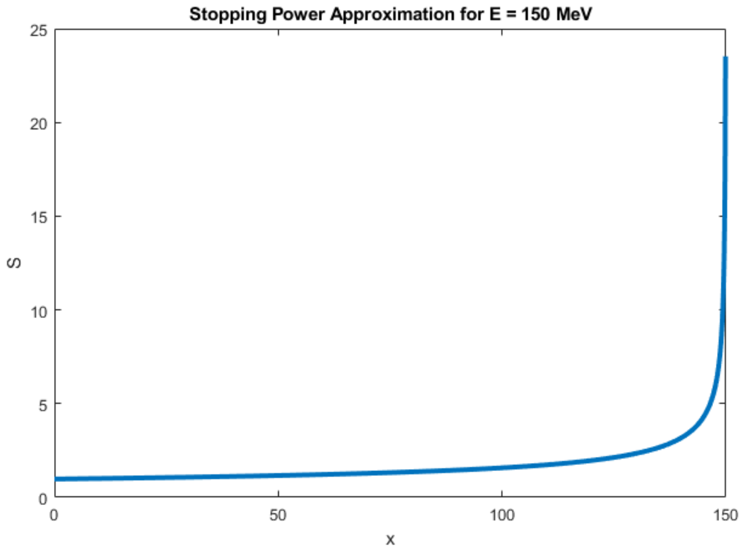
- CSDA Approximation for $E(x)$ with „remaining" range $(R - x)$:

$$E(x) = \frac{1}{\alpha^{\frac{1}{p}}} (R - x)^{\frac{1}{p}}$$

- Power-law approximation of stopping power:

$$S(x) = -\frac{dE}{dx} = \frac{(R - x)^{\frac{1}{p}-1}}{p\alpha^{\frac{1}{p}}}$$

→ Singularity!



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Fluence reduction:

- Get some inspiration from the photon world:

$$\Phi(x) = \Phi_0 e^{-\mu x}$$

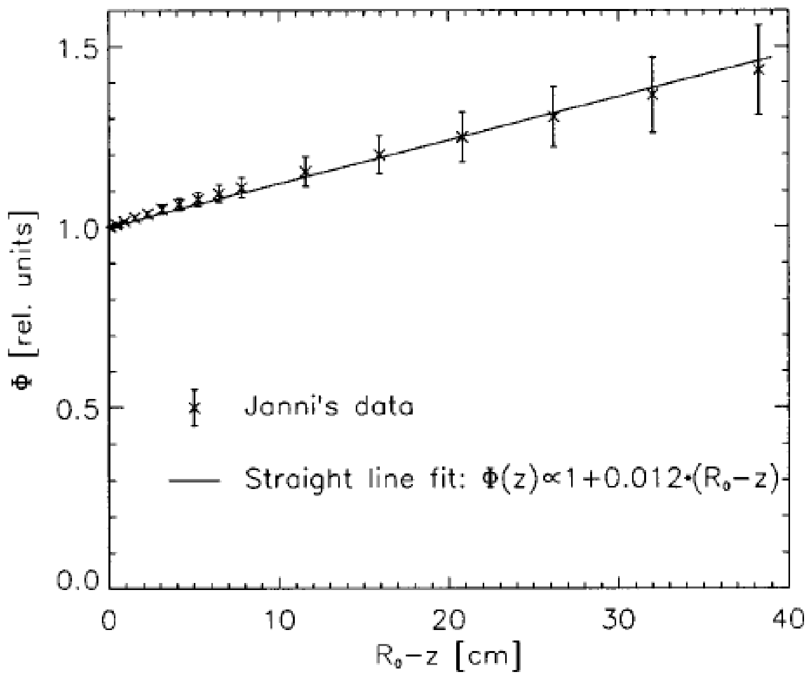
- For nuclear interactions: $\mu = \frac{1}{\lambda} = \rho_N \sigma_{abs}$
  → mean free path length $\lambda \approx 10 - 100\ cm$
  → quite unlikely event.

- For therapeutic ranges almost a straight line!
  → Alternative Fit:

$$\Phi(x) = \Phi_0 \frac{1 + \kappa(R - x)}{1 + \kappa R}$$

$$-\frac{d\Phi(x)}{dx} = \Phi_0 \frac{\kappa}{1 + \kappa R}$$

  - $\kappa \approx 0.012\ \mathrm{cm}^{-1}$



Bortfeld (1996)

Let‘s put everything together...

- Remember: Dose along incoming direction x:

$$D(x) = -\frac{1}{\rho}\frac{d\Psi}{dx} = -\frac{1}{\rho}\left(\Phi(x)\frac{dE}{dx} + \gamma\frac{d\Phi}{dx}E(x)\right)$$

- And we get:

$$\widehat{D}\,(x) = \frac{\Phi_0}{\rho}\frac{e^{-\mu x}}{\alpha^{\frac{1}{p}}}\left[\frac{1}{\mathrm{p}}(\mathrm{R}-\mathrm{x})^{\frac{1}{\mathrm{p}}-1} + \gamma\mu(R-x)^{\frac{1}{p}}\right]$$

- Why the „hat“?
- What about the singularity at R?

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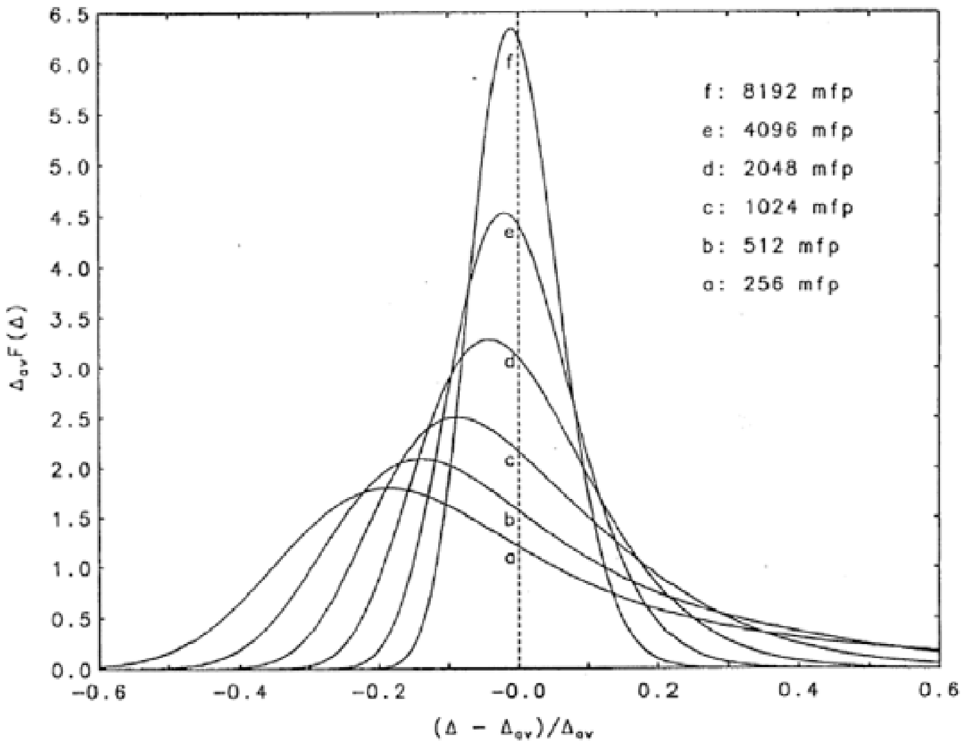
(Range) straggling

- Particle not continuously slowed down!
  → $\frac{dE}{dx}$ (nearly) follows a Landau distribution

  Mathematical plot twist / brain-teaser:
  What‘s the mean and variance of a Landau-distributed random variable?

- The thicker the absorber, the more we can approximate the distribution of energy loss with a Normal distribution:

$$f\left(\frac{dE}{dx}\right) \approx \mathcal{N}\left(\frac{dE}{dx}; \left\langle\frac{dE}{dx}\right\rangle, \sigma^2_{\frac{dE}{dx}}\right)$$



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Let‘s add the straggling...

- We can do a convolution:

$$D\,(x) = \langle\widehat{D}\rangle(x) = \int_0^R \widehat{D}(\bar{z})\,\mathcal{N}(z;\bar{z},\sigma_z^2)d\bar{z}$$

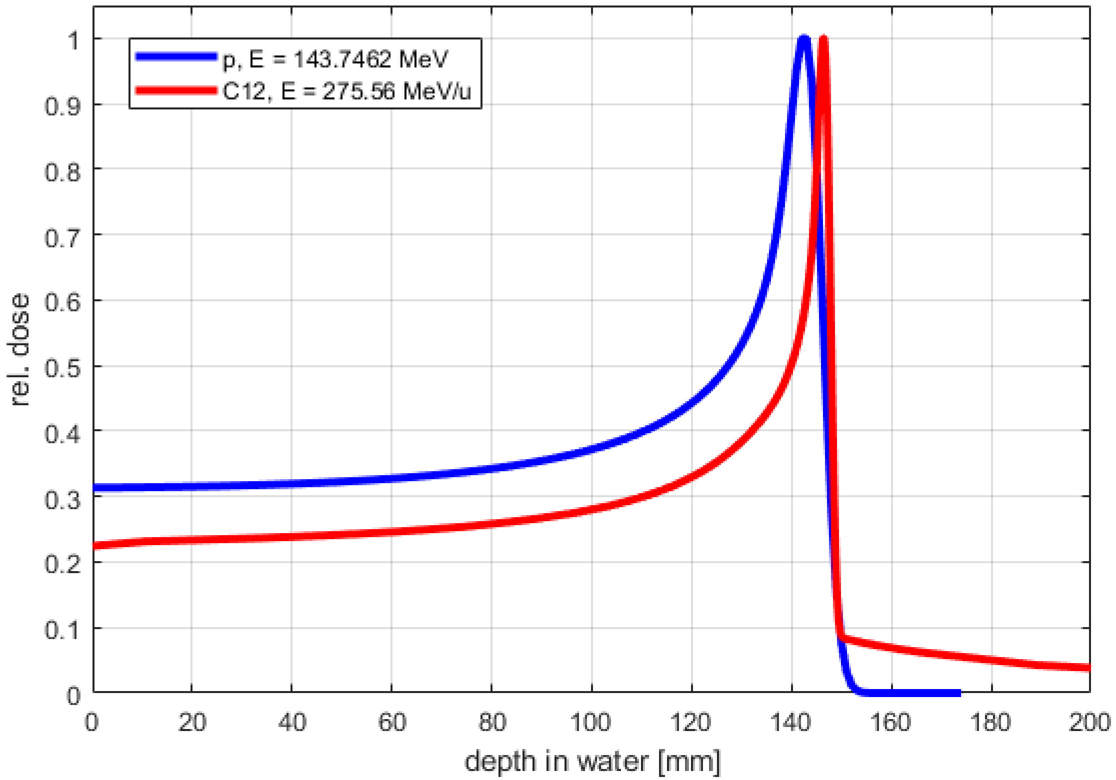
- This assumes that we „straggle“ both contributions (nuclear and Coulomb) since nuclear is neglegible

- One can approximate the straggling: $\sigma^2 \approx 0.0134R^{0.951}$

- Analytical solution possible, but tedious (involves parabolic cylinder functions, see Bortfeld 1996)

- Further straggling effects can be incorporated by larger σ via e.g. $\sigma_{new} = \sqrt{\sigma^2 + \sigma_{add}^2}$

… and obtain an analytical (proton) depth dose curve



- Does it work for other particles too?
- Yes, but: Fragmentation of particles A > 1 in nuclear interactions
- Range of fragments:
  - Assume $E_{primary} = E_{secondary}$
    → $R_s = \frac{A_s}{A_p}\frac{Z_p^2}{Z_s^2}R_p$
  - Example: C12 beam, p fragment
    → $R_{proton} = \frac{1}{12}\frac{6^2}{1^2}R_{carbon} = 3R_{carbon}$
- Fragments induce dose tail behind range

Summary of the last lectures and some additional info

1. The Bragg-peak can be (analytically) modeled using the CSDA
   → In practice, the Bragg-peak is measured/simulated and tabulated

2. Lateral scattering can be „okayishly“ approximated using a normally distributed scattering angle
   → The lateral distribution has nearly the shape of a widening Gaussian curve
   → Large-angle halos are in practice often modeled with additional Gaussians

3. We can model everything in water and then re-scale to the patient by using water equivalent depths
   - HU to relative stopping power conversion → not without uncertainties
   - Ray-casting through the image

→ Analytical pencil-beam dose calculation

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